A Mathematical Model of Stress Fracture in Human Bone

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Introduction to Bone Adaptation

- When a bone is exposed to a mechanical stimulus, such as everyday daily loading, its mechanical properties can change.
- A German anatomist, Julius Wolff, suggested this in the late 1800s.
- Wolff’s Law:
  - Every change in the form and function of a bone or of their function alone is followed by certain definite changes in their internal architecture, and equally definite secondary alterations in their external confirmation in accordance with mathematical laws.
Background Information

- Bone remodeling has 2 phases:
  - resorption → osteoclasts remove bone
  - formation → osteoblasts deposit new collagen
- Around 20% of bone tissue is replaced every year throughout the skeleton
- The entire remodeling process typically takes from 4 to 8 months
Negative Effects of Bone Remodeling

- Accelerates progression to a stress fracture
- Chain reaction: increased remodeling → more porosity → decreases elastic modulus → increases strain → further increases damage formation
- ∴ Internal remodeling may make bone more susceptible to stress fracture
Factors Involved with Stress Fracture

Diagram:
- Load
  - Cortical Area
    - Stress
      - Porosity
        - Damage Formation
          - BAD LOOP
      - Elastic Modulus
        - Strain
          - Damage
            - GOOD LOOP
            - Remodeling
        - Periosteal Callus
          - Damage Removal
Equations Used in Model

\[ \sigma = \frac{F}{A} \]

\[ E = 20(1 - P)^3 \]

\[ \dot{P} = Q_R N_R - Q_F N_F \]

\[ Q_R = \frac{\pi R_C^2}{T_R} \]

\[ Q_F = \pi \left( R_C^2 - R_H^2 \right) \frac{T_F}{T_T} \]

\[ N_F = \int_{t-(T_R+T_I)}^{t-(T_R+T_I+T_F)} f_a(t)dt \]

\[ N_R = \int_{t-T_R}^{t} f_a(t)dt \]

\[ f_a = \frac{f_a_{\text{max}} f_a_{\text{min}}}{f_a_{\text{min}} + (f_a_{\text{max}} - f_a_{\text{min}}) e^{k_R f_a_{\text{max}} (D-D_0)/D_0}} \]

\[ D = D_F - D_R \]

\[ \dot{D}_F = k_D R_L \varepsilon^q \]

\[ \dot{D}_R = D f_a \pi R_C^2 F_S \]

\[ \varepsilon = \frac{\sigma}{E} \]

\[ D_{\text{total}} = D + D_0 \]

\[ P_{\text{total}} = P + P_0 \]
Inputs and Constants

- **Inputs**
  - $S = 300$ to $3050$ strain range, microstrain, $10^{-6}$ m/m
  - $R_L = 3000$ loading rate, cycles/day
  - $D_0 = .0366$ initial damage, crack length per unit section area, mm$^{-1}$
  - $F = 2614$ maximum applied force, N
  - $P_0 = .0448$ initial porosity

- **Constants**
  - $A = 500$ cross-sectional area of bone where force applied, mm$^2$
  - $R_H = .02$ completed Haversian canal radius, mm
  - $R_c = .095$ osteonal cement line radius, mm
  - $T_R = 24$ resorbing period, days
  - $T_I = 8$ reversing period, days
  - $T_F = 64$ refilling period, days
  - $F_s = 5$ damage repair specificity factor, dimensionless
  - $q = 4$ exponent on strain range, dimensionless
  - $k_D = 185000$ damage formation rate coefficient, day cycle$^{-1}$ mm$^{-1}$, microstrain$^{-q}$
  - $f_{amax} = 0.5$ maximum activation frequency value, BMUs mm$^{-2}$ day$^{-1}$
  - $f_{amin} = 0.0067$ minimum activation frequency value, BMUs mm$^{-2}$ day$^{-1}$
  - $k_R = -1.6$ coefficient for $f_a$ versus $D$ dose-response curve, mm$^2$ day BMU$^{-1}$
Simplified Simulink Model
Strain range 300 to 3050 (step 250)
Problems with the Model

- Porosity calculations not worked in
- Porosity affects elastic modulus, which affects strain, which affects damage formation, and so on
- Very complicated system → cannot predict stress fracture yet → only model specific situations
Future improvements of model

- Include porosity and periosteal callus
- Create another summation for damage formation
  - Then additional loading, such as an exercise program, could be simulated to see its effects
In Conclusion…

- Stress fracture is difficult to model because of the unique mathematical relationships between the various factors involved.
- It is even harder to predict since many biological factors are also involved, some as simple as age and sex, and others as complicated as previous activity level or lifestyle.
References


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