1) Consider simple diffusion on the interval 0<x<L subject to absorbing boundary conditions at x=0,L. Thus, the probability distribution of diffusing particles obeys the 1D Diffusion Equation:

\[
\frac{\partial p(x,t)}{\partial t} = D \frac{\partial^2 p(x,t)}{\partial x^2}
\]  

(1)

with \(p(0,t) = p(L,t) = 0\). [D is the appropriate diffusion constant.] Let the NxN matrix \(\Omega^{(N)}\) be defined as the banded matrix having -2 on the diagonal, 1 on the first band above and below the diagonal, and 0 elsewhere. For example, for N=4:

\[
\Omega^{(4)} = \begin{pmatrix}
-2 & 1 & 0 & 0 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -2 \\
\end{pmatrix}
\]

a) Discretize \(p(x,t)\) into an N-dimensional vector \((p_1, p_2, \ldots, p_N)^T\) such that \(p_j(t) = p(ja,t)\), where \(j = 1, 2, \ldots, N\) and \(a\) is the grid spacing, \(a = L/(N+1)\). Show that the discrete analog of Eq. 1 is:

\[
\dot{\mathbf{p}} = \frac{D}{a^2} \Omega^{(N)} \mathbf{p}
\]

(2)

b) Eq. 2 can be directly integrated to give:

\[
\mathbf{p}(t) = \exp\left(\frac{D}{a^2} \Delta^{(N)} t\right) \mathbf{p}(0)
\]

Setting \(L=1\) and \(N=21\), let \(p_1(0) = 1\) and all other components equal to 0. (This corresponds to placing a particle at the center of the box.) Calculate and plot the time evolution of the probability distribution.
c) All the eigenvalues of $\lambda_1^N$ are negative. Identify the least negative eigenvalue: call this $\lambda_1$ and the corresponding eigenvector $\vec{v}_1$. Show that the approximation

$$\vec{p}(t) \approx (\vec{v}_1)_{11} \vec{v}_1 e^{\lambda_1 \vec{r}}$$

becomes very accurate after short-time transients die off.

2) Given any analytic function $f(z) = u(x, y) + iv(x, y)$ of a complex variable $z = x + iy$, it can be shown that both $u$ and $v$ satisfy the 2D Laplace Equation. That is,

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \quad (3)$$

and likewise for $v$. [Roughly speaking, an analytic function is one which can be represented as a sum of integral powers of its argument.]

a) Pick an analytic function $f(z)$ (your choice!). Show that $u(x, y) = \text{Re}(f(z))$ satisfies the Laplace Eq. (3); also check that $v(x, y) = \text{Im}(f(z))$ satisfies the same equation.

b) Pick some function that satisfies the 2D Laplace Eq. [e.g., based on part a]): denote this as $u(x, y)$. Pick a rectangular perimeter in the $x$-$y$ plane (again, your choice).

i) Using the known values of $u$ on the perimeter, use the Mathcad subroutine relax to compute an approximate solution to the Laplace Eq. in the interior region. (The linear discretization index $N$ is up to you, but check for convergence as described below.)

ii) Make a contour plot of the function computed using relax in part i). Compare this to the exact analytical solution obtained in part a). Show that as $N$ is increased, the agreement between the numerical and analytical solutions for $u(x, y)$ improves. (To see the convergence process more clearly, it may be useful to plot $u(x, y_f)$ vs $x$, where $y_f$ is a fixed value of $y$ in the interior region.)